MAT 534: HOMEWORK 12
DUE W NOV 18

Problems marked by asterisk (*) are optional.

Throughout this assignment, all vector spaces are over a field F.

1. Let T be a linear operator on the finite-dimensional space V. Suppose there is a linear operator U on V such that $TU = I$. Prove that T is invertible, i.e. has both left and right inverse, and $U = T^{-1}$.

Show that this is false when V is not finite-dimensional. (Hint: Let $T = D$ be the differentiation operator on the space of polynomials.)

2. Let V_1 and V_2 be finite-dimensional subspaces of a vector space V. Verify that $V_1 \cap V_2$ and $V_1 + V_2$ are also subspaces.

(a) Prove that

$$\dim_F(V_1 + V_2) = \dim_F V_1 + \dim_F V_2 - \dim_F(V_1 \cap V_2).$$

(b) If V is finite-dimensional, prove that it is possible to choose a basis $\{v_i\}_{i\in I}$ in V and two subsets $I_1, I_2 \subset I$ such that $\{v_i\}_{i\in I_1}$ is a basis of V_1, $\{v_i\}_{i\in I_2}$ is a basis of V_2, and $\{v_i\}_{i\in I_1\cup I_2}$ is a basis of $V_1 + V_2$.

3. Let $P : V \to V$ be a linear operator on a finite-dimensional space V such that $P^2 = P$. Prove that $V = V_1 \oplus V_2$, where $P|_{V_1} = \mathrm{id}$ and $P|_{V_2} = 0$, so that P is a projection operator.

4. Let A, B be commuting linear operators on V such that $A^2 = A$, $B^2 = B$. Prove that then $\ker(AB) = \ker(A) + \ker(B)$.

***5.** Prove the formula for Cauchy and Vandermonde determinants.

6. Let

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ b_1 & a_1 & a_1 & \cdots & a_1 \\ b_1 & b_2 & a_2 & \cdots & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_1 & b_2 & b_3 & \cdots & a_n \end{pmatrix}.$$

Compute $\det A$.

7. Let A be $n \times n$ matrix with 2 on the diagonal, -1 immediately below and above the diagonal and zeros elsewhere. Compute $\det A$.

***8.** Let A be $m \times m$ matrix and B be $n \times n$ matrix. Prove that

$$\det(A \otimes B) = \det A^n \det B^m.$$